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Rank Reduction of Ill-Conditioned Matrices in Waveguide Junction Problems

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Abstract—**A new low-rank spectral expansion technique for solving the ordinarily intractable matrix equations obtained from waveguide field equivalence theorem decompositions is described. The method facilitates the analysis of waveguide discontinuity problems that resist ordinary methods of solution. The technique is illustrated for the problem of scattering at a slant interface in a rectangular waveguide.**

I. INTRODUCTION

THE integral equations, and the corresponding matrix equations, that represent scattering at a waveguide discontinuity often exhibit ill-conditioned behavior. This results in computational difficulties as inversion of such matrices is inaccurate for even large-order truncated versions of the matrix. It is shown here, however, that it may be possible to take advantage of the often relatively low effective rank of the ill-conditioned portion of the matrix to overcome such difficulties.

In the following, a typical problem, that of scattering at a waveguide discontinuity, is solved by developing equations that are exact but ill conditioned. First, field equivalence theorems are used to reduce the structure to two uniformly filled waveguides with equivalent electric and magnetic current sheets at the discontinuity surface. Integral equations for the current sheets are then derived, using the null field condition in the two simpler waveguide structures. By writing series expansions for the current sheets, the integral equations are reduced to a system of linear algebraic equations for the current expansion coefficients. These exact equations are asymptotically ill conditioned. By a low rank spectral decomposition of the matrix representing the ill-conditioned portion of the equations, it is possible to solve for the currents without

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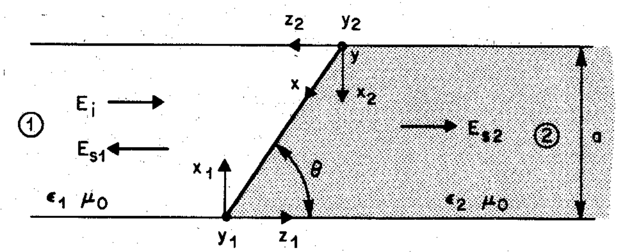

Fig. 1. Top view of rectangular waveguide traversed by slanted interface between two media.

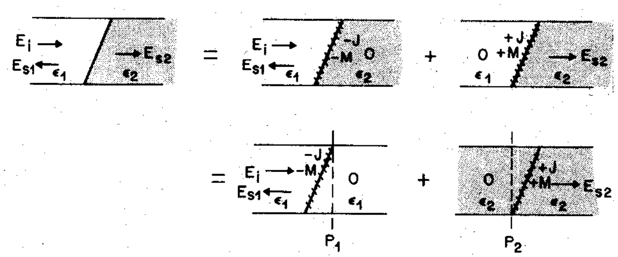

Fig. 2. Equivalence theorem decomposition into two waveguides with equivalent sources at the interface. The waveguides can be considered to be filled homogeneously.

inverting a large ill-conditioned matrix, while not ignoring high-order terms. The scattered fields are then readily obtained from the currents.

The new technique of matrix spectral expansion is illustrated by solving the problem of scattering at a slant interface in a rectangular waveguide. This problem has recently received attention in the literature. Chow and Wu [1] solve the problem by using a moment method with mixed basis functions. De Jong and Offringa [2] use the Green's function for the waveguide to derive integral equations for the fields at the interface. Kashyap [3] solves for the scattered fields by using a geometrical optics approximation for the field on the interface. In the analysis presented here the waveguide structure shown in Fig. 1 is decomposed, via field equivalence theorems, into two simpler structures that are easier to analyze. A large-order ill-conditioned matrix is obtained in the analysis. The inversion of this matrix is reduced through spectral decomposition to the inversion of a well-conditioned low-order matrix.

II. Theory

Fig. 1 shows a top view of a rectangular waveguide whose broad dimension is traversed by a slanted interface between two media. The structure is considered to be two waveguides joined at the slanted plane. Each waveguide has its own coordinate system, as shown. The structure has no variation with respect to y_1. The modes for the waveguides on each side of the interface are expressed by $\boldsymbol{E}_{lm}^{\pm}$, $\boldsymbol{H}_{lm}^{\pm}$, where l is the waveguide region number (1 or 2) and m is the mode number.

The incident field is given by the modal amplitude coefficients a_{1m} for waveguide 1 as

$$\boldsymbol{E}_i = \sum_{m=1}^{\infty} a_{1m} \boldsymbol{E}_{1m}^{+}(x_1,y_1,z_1) \tag{1}$$

while the scattered fields are given by the scattered modal amplitude coefficients b_{lm} in both waveguides as

$$\boldsymbol{E}_{sl} = \sum_{m=1}^{\infty} b_{lm} \boldsymbol{E}_{lm}^{-}(x_l,y_l,z_l). \tag{2}$$

The solution of this reflection–transmission problem for the scattering coefficients is considerably more complex than that of excitation of the homogeneous waveguides by given current sources, but the former can be reduced to the latter by use of field equivalence theorems.

An exhaustive development of field equivalence theorems appears in the literature [4]–[14]. Love's equivalence theorem [6], or Huygen's principle [9]–[12], allows the replacement of the discontinuity problem on the left side of Fig. 2 by the superposition of the pair of configurations on the right side, much like Thevenin's theorem replaces portions of a network by equivalent sources. In each of the pair, the fictitious electric and magnetic current sheets on the interface, $\boldsymbol{J}$ and $\boldsymbol{M}$, must be equal to the discontinuities in the tangential fields, from the actual fields on one side to zero on the other. A superposition of these two configurations and fields clearly yields the original one. Considering the equivalent current sheets in each problem on the right side of the figure as sources, they are seen to radiate fields on one side that are the same as those actually scattered by the discontinuity, but the two current sheets also radiate a field that exactly cancels the total field on the other side.

The importance of the null field regions of the two structures on the right side lies in that they may be replaced with any media whatsoever without affecting the radiated fields. Specifically, the null field regions can be replaced with the same media as on the other sides of the equivalent currents, as shown in the figure. The solution for the fields of the original structure has thus been made equivalent to that for the two homogeneous waveguide problems shown, except that the current sheets are not yet known. The sources $\boldsymbol{J}$ and $\boldsymbol{M}$ are to be found from the condition that they produce the null fields to one side in each problem. Once $\boldsymbol{J}$ and $\boldsymbol{M}$ are known, the scattered fields may be found in terms of them.

To derive equations for $\boldsymbol{J}$ and $\boldsymbol{M}$ it is sufficient to require them to radiate null fields to the right of plane P_1 and to the left of plane P_2. This further simplification is valid because there are no sources or discontinuities between the current sheets and the planes P_1 or P_2, so that the simplified condition still assures a null field throughout the entire region. The positions of P_1 and P_2 are not unique, but are conveniently chosen outside the sourced region and transversely across the waveguides so that the null field condition is satisfied by requiring all modal amplitudes radiated away from the currents into the null field regions to vanish at these planes. These constraints represent an extended boundary condition for the problem [15], [16].

The modal amplitudes generated by the equivalent currents and by the incident field are readily obtained for homogeneous waveguides by using the Lorentz reciprocity theorem [5]. Setting these amplitudes equal to zero yields the following equations for the current sheets:

$$-\frac{1}{P_{1m}}\int_S [E_{1m}^- \cdot (-J) - H_{1m}^- \cdot (-M)]\, dS + a_{1m} = 0$$
$$-\frac{1}{P_{2m}}\int_S [E_{2m} \cdot (+J) - H_{2m}^- \cdot (+M)]\, dS = 0 \qquad (3)$$

where the integrations span the surface of discontinuity and the normalizing modal power factor is given by the known integrals of the normal modes over the waveguide cross sections

$$P_{lm} = 2\int_{S_l} e_{lm} \times h_{lm} \cdot dS.$$

The scattered fields may be found once J and M are known by again applying the Lorentz reciprocity theorem.

$$b_{1m} = -\frac{1}{P_{1m}}\int_S [E_{1m}^+ \cdot (-J) - H_{1m}^+ \cdot (-M)]\, dS$$
$$b_{2m} = -\frac{1}{P_{2m}}\int_S [E_{2m}^+ \cdot (+J) - H_{2m}^+ \cdot (+M)]\, dS. \qquad (4)$$

One way of handling the integral equations (3) for the unknown current sheets J, M is to convert them to simultaneous linear equations for the coefficients of series expansions for the unknowns. Upon selecting any suitable set of basis functions $J_n(x)$, $M_n(x)$, series expansions may be written for J and M. Substitution of these expansions into (3) then yields a system of linear algebraic equations for the current expansion coefficients j_n and m_n, as in

$$J(x) = \sum_{n=1}^{\infty} j_n J_n(x)$$
$$M(x) = \sum_{n=1}^{\infty} m_n M_n(x). \qquad (5)$$

The resultant system of linear algebraic equations for the current expansion coefficients can be written in matrix form as

$$Gc = s \qquad (6)$$

where c comprises the current expansion coefficients, s has the incident modal amplitude coefficients, and the infinite-order matrix G can be partitioned as

$$G = \begin{bmatrix} G_{1e} & G_{1h} \\ G_{2e} & G_{2h} \end{bmatrix} \qquad (7)$$

with components

$$G_{1e}(m,n) = \int_S \frac{E_{1m}^- \cdot J_n}{P_{1m}}\, dS \qquad (8)$$

$$G_{1h}(m,n) = -\int_S \frac{H_{1m}^- \cdot M_n}{P_{1m}}\, dS \qquad (9)$$

$$G_{2e}(m,n) = \int_S \frac{E_{2m}^- \cdot J_n}{P_{2m}}\, dS \qquad (10)$$

$$G_{2h}(m,n) = -\int_S \frac{H_{2m}^- \cdot M_n}{P_{2m}}\, dS. \qquad (11)$$

The unknown current coefficient vector c is then given in terms of the incident modal amplitude vector s by

$$c = G^{-1}s \qquad (12)$$

which formally requires the inversion of the infinite-order matrix G.

The choice of current expansion functions in (5) is arbitrary but affects the convergence of the infinite sums. To arrive at a useful choice, note from the equivalent problems in Fig. 2 that J and M will be proportional, respectively, to magnetic and electric fields tangential to the discontinuity surface. For the purpose of illustrating the solution technique, assume that only TE_{n0} modes are incident. Then since the structure has no y variation, only TE_{n0} modes will be scattered. The electric field is along $\hat{y}$ and therefore M is along $\hat{x}$. The magnetic field is in the plane orthogonal to $\hat{y}$ and therefore J is along $\hat{y}$. The electric field vanishes at the waveguide walls at $x = 0$ and $x = a/\sin\theta$. It is thus reasonable to choose

$$M_n(x) = \sin[n\pi(x/a)\sin\theta]\hat{x} \qquad (13)$$

where n is a positive integer. The magnetic field does not vanish at the walls. However, the modes used to expand the field do have a sinusoidal dependence along the discontinuity. Thus choose

$$J_n(x) = \cos[n\pi(x/a)\sin\theta]\hat{y} \qquad (14)$$

where n is a positive integer or zero. Note that this choice of expansion functions is by no means the only choice, nor necessarily the best choice. Another reasonable choice might be the forward- and reverse-going modes of either or both waveguides on each side of the discontinuity. The analysis below demonstrates that the ill conditioning that may arise from injudicious choices of expansion functions can be dealt with effectively by rank reduction.

Substitution into the integrals for G yields

$$G_{1e}(m,n) = (-1)^m e^{j\Phi}(aY_{1m})^{-1/2} I_2(m,n,-\beta_{1m}) \qquad (15)$$

$$G_{1h}(m,n) = (-1)^m e^{j\Phi}(Y_{1m}/a)^{1/2} \cdot [\sin\theta I_1(m,n,-\beta_{1m}) - j(p_{1m}/\beta_{1m})\cos\theta I_2(n,m,-\beta_{1m})] \qquad (16)$$

$$G_{2e}(m,n) = (aY_{2m})^{-1/2} I_2(m,n,+\beta_{2m}) \qquad (17)$$

$$G_{2h}(m,n) = -(Y_{2m}/a)^{1/2} \cdot [\sin\theta I_1(m,n,+\beta_{2m}) + j(p_{2m}/\beta_{2m})\cos\theta I_2(n,m,+\beta_{2m})] \qquad (18)$$

where Y_{lm} is the modal admittance, $\Phi = \beta_{1m}a\cot\theta$, p_{lm} is the cutoff wavenumber for the mth mode, and

$$I_1(m,n,\beta) = \int_0^{a/\sin\theta} \sin[m\pi(x/a)\sin\theta] \cdot \sin[n\pi(x/a)\sin\theta] e^{j\beta x\cos\theta}\, dx \qquad (19)$$

$$I_2(m,n,\beta) = \int_0^{a/\sin\theta} \sin[m\pi(x/a)\sin\theta] \cdot \cos[n\pi(x/a)\sin\theta] e^{j\beta x\cos\theta}\, dx. \qquad (20)$$

The integrals may readily be evaluated in closed form, with the following results:

$$I_1(m,n,\beta) = j(\beta/2)\cos\theta R_{mn}[S_{mn}^{+} - S_{mn}^{-}] \quad (21)$$

$$I_2(m,n,\beta) = (\pi/2a)\sin\theta R_{mn}[T_{mn}^{+} + T_{mn}^{-}] \quad (22)$$

where

$$R_{mn} = 1 - (-1)^{m+n}e^{j\beta a \cot\theta} \quad (23)$$

$$S_{mn}^{\pm} = \{[(m \pm n)(\pi/a)\sin\theta]^2 - (\beta\cos\theta)^2\}^{-1} \quad (24)$$

$$T_{mn}^{\pm} = (m \pm n)S_{mn}^{\pm}. \quad (25)$$

The infinite-order matrix G remains to be inverted, but truncated versions of this matrix are found to be ill conditioned and require special treatment.

III. Ill-Conditioned Matrices

Because of the behavior of the matrix elements for large values of m or n, the matrix G is ill conditioned. This is traceable to the properties of the integrals I_1 and I_2. For evanescent modes, $\beta_{lm} = -j\alpha_{lm}$ approaches $-j(m\pi/a)$ for large m. Except for the special case of $\theta = 90°$, the very steeply rising exponential envelope which multiplies the two sinusoids in the integrands prevents the rapid decay of the integrals as m and n get larger and more separated from each other. In addition, the value of I_1 or I_2 will not vary much within a small range of large m or n, since the exponential factor in the integrand counteracts the tendency of the sinusoidal factors to be orthogonal.

To solve (6) exactly for the unknown current coefficient vector, G must be inverted. The direct inversion of this matrix, or of finite-order truncated versions of G, is not practical, especially for large order, because of its ill conditioning. However, as will be shown, a low-rank spectral decomposition can be applied to G to avoid the inversion of a large-order matrix.

We rearrange and partition G to rewrite $Gc = s$ so as to segregate low-order modes and expansion functions from the presumably less important high-order ones, in the form

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix}\begin{bmatrix} p \\ q \end{bmatrix} = \begin{bmatrix} r \\ 0 \end{bmatrix} \quad (26)$$

not as in (7), which separated electric and magnetic components for purposes of evaluating G. A is a square matrix obtained by keeping only the first few low-order terms in the expansion for both J and M and requiring that only the first few modes of waveguides 1 and 2 satisfy the null-field condition at planes P_1 and P_2. D is an infinite-order ill-conditioned submatrix, corresponding to only high-order modes and expansion functions; B relates low-order modes to high-order expansion functions, and vice versa for C. The vector of current expansion coefficient c is correspondingly partitioned into p, containing the first few coefficients, and q for the higher order ones. The known incident modal amplitude vector s is partitioned as r and the null vector since, for a single low-order incident mode, s and also r have only one nonzero element. For more general excitations, s may be partitioned into two nonzero parts and the solution readily extended to include the additional sources.

One approach to the inversion problem is to ignore higher order terms entirely. If G is severely truncated to merely A, a crude solution p' for the first few unknown current expansion coefficients p is given by

$$p' = A^{-1}r. \quad (27)$$

This is to be compared with the exact solution for all the unknown current expansion coefficients, expressible in terms of the partitions A, B, C, D as

$$p = (A - BD^{-1}C)^{-1}r$$
$$q = -D^{-1}C(A - BD^{-1}C)^{-1}r. \quad (28)$$

If even an approximate inverse of submatrix D were available, (28) would provide solutions for p, q that do not completely ignore the high-order modes, as does (27). An obvious approach is to truncate D. However, since D is ill conditioned, D^{-1} cannot be readily evaluated, especially when D is of high order. The desired improvement to the crude solution given by (27) may nevertheless be obtained, by spectrally decomposing D.

The spectral decomposition of D, truncated to an $N \times N$ matrix, is given by [17]

$$D = \sum_{i=1}^{N} \lambda_i e_i u_i^T \quad (29)$$

in terms of its eigenvalues λ and orthonormal eigenvectors e, u^T; T denotes transposition. D is ill conditioned but not singular. Now assume that a good approximation for D is obtained by using only the first K largest eigenvalues in the expansion (29), with $K < N$. K is to be the approximate rank of D. If K were the exact rank of D, then K eigenvalues would exactly represent D. Accordingly, an approximate expression for D may be written as

$$D = fg \quad (30)$$

where the matrices f, g are formed from the eigenvectors of D:

$$f = [\lambda_1 e_1, \lambda_2 e_2, \cdots, \lambda_K e_K]$$
$$g = [u_1, u_2, \cdots, u_K]^T. \quad (31)$$

The matrices f and g are $N \times K$ and $K \times N$, respectively. Within the approximation that q is in the space spanned by matrix f, substitution of (30) into (26) leads to the solution

$$p = [A - Bf(gf)^{-2}gC]^{-1}r$$
$$q = -f(gf)^{-2}gC[A - Bf(gf)^{-2}gC]^{-1}r. \quad (32)$$

This result accounts for the high-order modes, to the extent that they can be expanded in the K retained eigenvectors of D.

Note that neither the ill-conditioned matrix D nor its singular approximation fg needs to be inverted for this solution. The problem of inverting the large-order matrix D has been reduced to that of inverting a much lower order matrix gf, which is $K \times K$ instead of $N \times N$ and is not ill conditioned.

There are numerous standard methods which may be used to find the eigenvalues and eigenvectors of D needed in the spectral decomposition. One method transforms D to an upper Hessenberg matrix and then computes the eigenvalues using the QR double-step procedure. The eigenvectors are then found by performing inverse iteration on the Hessenberg matrix [18]–[20]. There also exist various iterative schemes which find only the first K largest eigenvalues and corresponding eigenvectors [18], [21].

The quantity $D^- = f(gf)^{-2}g$ appearing in (32) is the group inverse of D [22]. The group inverse is one of several pseudoinverses that could be used in this context. Another one is generated by the singular-value decomposition [22]–[24], which yields the minimum-norm least squares inversion. It is closely related to the following modification, designed to enhance the ratio of magnitudes of the K retained eigenvalues to those of the $N - K$ neglected ones. If the second equation in (26) is premultiplied by D^+, where D^+ is the complex conjugate transpose of D, there results

$$\begin{bmatrix} A & B \\ D^+C & D^+D \end{bmatrix} \begin{bmatrix} p \\ q \end{bmatrix} = \begin{bmatrix} r \\ 0 \end{bmatrix}. \tag{33}$$

Observe that D^+D is a Hermitian matrix. This matrix equation is of the same form as (26) and the same analysis is applicable in solving for $\boldsymbol{p}$ and $\boldsymbol{q}$. For Hermitian matrices factored as in (30), with gf nonsingular, the group inverse becomes the Moore–Penrose inverse. A solution to (33) obtained using this inverse involves a minimum-norm least square solution for $\boldsymbol{q}$ in terms of $\boldsymbol{p}$. This solution can be shown to be unique. It always exists even though the equation being solved may be inconsistent or have families of solutions [22]–[24]. A disadvantage of this modification is the reduced numerical accuracy of D^+D compared to that of D. This can be overcome by applying the singular-value decomposition directly to the non-Hermitian matrix D [25].

The accuracy of the solution is related to that of the factorization approximation of D or D^+D. In order to have a good rank-K approximation for D or D^+D, the first K eigenvalues should have much larger magnitudes than the $N - K$ remaining ones. Even if this condition is met, the approximate solution for $\boldsymbol{p}$ and $\boldsymbol{q}$ may still be significantly different from the exact, unattainable solution. This would occur if $D^+C\boldsymbol{p}$ were to have significant components along any of the $N - K$ neglected eigenvectors of D^+D, or if $\boldsymbol{r}$ could not be resolved in the space spanned by $A - B(D^+D)^-C$ in (32). However, the quantities ultimately to be determined are the scattered mode amplitudes of (4). Thus an examination of the scattered fields will indicate whether the solution for the current expansion coefficients has sufficient accuracy. The scattered fields should satisfy basic physical principles such as power conservation and reciprocity. A further discussion of the accuracy of the procedure appears below in connection with some numerical results.

IV. Numerical Results

The analysis of Sections II and III has been applied to the waveguide slant discontinuity shown in Fig. 1, with $\theta = 45°$. The relative dielectric constants of the media filling the waveguides on the left and right sides of the discontinuity were taken as $\varepsilon_1 = 1$ and $\varepsilon_2 = 3$, respectively. For a unit-amplitude TE_{10} mode incident from $z_1 = -\infty$, the scattered amplitudes of the propagating modes on both sides of the discontinuity were computed for $1 < f/f_c < 1.732$, where f_c is the TE_{10} cutoff frequency in that waveguide in which the TE_{10} mode is incident. At the highest frequency considered, both the TE_{10} and TE_{20} modes propagate to the right of the discontinuity, but only the TE_{10} mode propagates to the left.

The solution procedure can be summarized as follows. Based upon known solutions to the scattering problem, truncating G in (6) to a 16×16 matrix is essentially equivalent to leaving it of infinite order. Examination of the portion of G corresponding to large m and n reveals that it is approximately of rank 4; the first four eigenvalues of D^+D have much larger magnitudes than the remaining ones. For example, for $f/f_c = 1.1$, the four largest eigenvalues were 0.660, 0.539, 0.0284, and 0.018. The fifth eigenvalue was 0.000843, and the remaining ones were all much smaller than this one. These eigenvalues were computed using an iterative technique [18]. If the approximation is made that D^+D is exactly rank 4, then gf in (32) will be 4×4. In (26) or (33) A can conveniently be made 4×4, so that the largest matrix that needs to be inverted is only 4×4.

The magnitudes of the reflected and transmitted propagating modes are given in Fig. 3. Fig. 4 shows the total power carried by the scattered modes. Three sets of solutions are shown in the figures. One set is for the severely truncated equations represented by (27). Another set is for the spectrally decomposed D^+D matrix with solutions for $\boldsymbol{p}$ and $\boldsymbol{q}$ as represented by (32). The third set of solutions is the exact one obtained either from De Jong and Offringa [2] or from (28) by directly inverting D. The results obtained by inverting only A disagree with the correct ones by as much as a factor of 2 and power conservation is violated by up to 30 percent. However, the errors of the corrected results, assuming D^+D is rank 4, are of the order of 5 percent, no worse than 20 percent, and power conservation is satisfied to within 4 percent. This represents a significant improvement over the severely truncated case.

A limitation to the analysis occurs if the frequency becomes too high or if the dielectric constant discontinuity becomes too great. Then the equivalent current sheets are expected to have many variations along the discontinuity plane, and a large number of terms may be needed in the expansions for the current sheets given by (5). It might then be appropriate to choose a better set of current expansion functions, or perhaps to use a geometrical optics approximation for the currents.

The accuracy or reliability of the solution process is not predictable in general but is subject to certain numerical checks. Besides monitoring how well power conservation

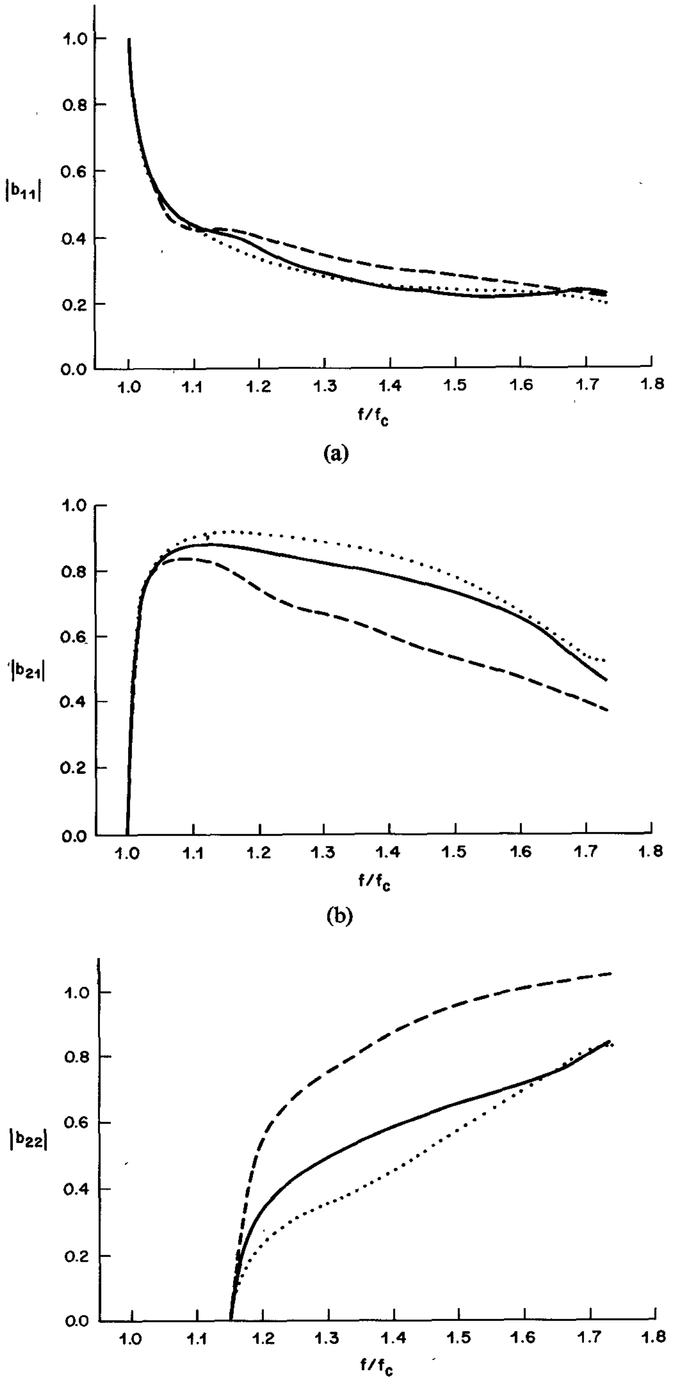

Fig. 3. (a) Magnitude of reflection coefficient for TE_{10} mode. (b) Magnitude of transmission coefficient for TE_{10} mode. (c) Magnitude of transmission coefficient for TE_{20} mode. The exact solution is given by a dotted line, the severely truncated solution by a dashed line, and the low-rank correction by a solid line.

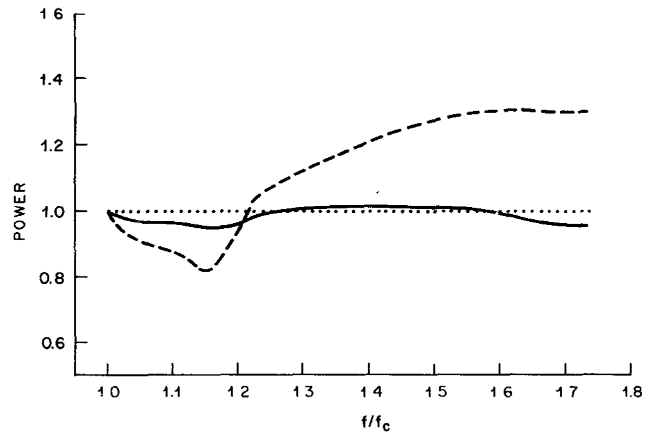

Fig. 4. Total scattered power as fraction of incident power. The exact solution is given by a dotted line, the severely truncated solution by a dashed line, and the low-rank correction by a solid line.

and reciprocity are satisfied, the convergence of the results with increasing values of the reduced rank K and with increasing order of the truncated matrix A can and should be verified. Comparisons were made with the solutions obtainable by direct inversion of large-order versions of the G matrix. Typically, a rank-reduced 4×4 inversion agreed with a direct inversion of a 16×16 matrix to within 5 percent. It should be noted that, in accordance with the variational principle that governs the scattering calculations [26], the results for the reflection and transmission coefficients are more accurate than those for the equivalent current sources.

V. Summary

The problem of scattering at a waveguide discontinuity was solved by using field equivalence theorems to reduce the structure to two uniformly filled waveguides with equivalent electric and magnetic current sheets at the discontinuity surface. Integral equations for the current sheets were derived using the null-field condition in the two simpler homogeneous structures. By writing series expansions for the current sheets, the integral equations were converted to a linear system of algebraic equations for the current expansion coefficients. The equations were found, however, to be asymptotically ill conditioned. By spectrally decomposing the matrix representing the ill-conditioned portion of these equations, and retaining only the major eigenvalues, it was possible to obtain a close approximation to the currents without inverting a large ill-conditioned matrix. The scattered fields were then readily obtained from the currents. The new technique of using a low-rank matrix spectral decomposition to solve the ordinarily intractable equations obtained from the field-equivalence-theorem waveguide decomposition facilitates the analysis of problems that resist ordinary methods of solution. The technique was illustrated for the problem of scattering at a waveguide slant, with results in agreement with other available solutions to within 5 percent typically.

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The New Similarity Rules Applied to Argon Microwave Noise Sources

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***Abstract*—It is shown that when the noise temperatures of argon plasma noise generators, operated at fixed current/radius ratios, are plotted as $1/T_N$ versus ln (pr), the experimental data form a straight line.**

Introduction

THE purposes of this paper are to show that the noise temperatures of commercial argon noise sources agree when a comparison is made based upon the new similarity rules which require scaling at constant current/radius ratios, and to show that the data obey a relationship of the form $1/T_N \propto \ln (pr)$, which permits using a linear least squares fit of experimental data.

Historical Microwave Developments

Beginning with the invention of the plasma noise source by Mumford in 1949 [1], many workers measured the noise temperatures of plasmas. An extensive experimental investigation and summary of previously published data was made by Olson in 1968 [2]. His conclusions were that the noise temperatures measured under various conditions did not satisfactorily agree, and that they did not agree with the von Engel and Steenbeck theoretical value [3, p. 86], [4, p. 242].

Later, it was discovered that a close agreement appeared to exist between his data and that taken by Denson and Halford [5] when only data from wall-contained plasmas were considered; this was shown in [6, fig. 1]. Comparisons between the noise temperatures obtained from different radius tubes were *always* made by invoking the traditional similarity rules [4, p. 288], [7, p. 209], [8, p. 59] which required that the current be the same in all tubes, independent of the radii.

The New Similarity Rules

In 1969 a new set of similarity laws requiring scaling at constant current/radius ratios was formulated by Pfau *et al.* [9]. Unfortunately, the new rules were not immediately widely known, and they were "rediscovered" at least twice since then. In 1975 it was shown in [10, fig. 4] that the new

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